$\boxed{\text{Example}}$

(1) $\log(1+\sqrt{3}i) = \ln|1+\sqrt{3}i| + i\arg(1+\sqrt{3}i)$
$= \ln 4 + i(\pi/3 + 2K\pi)$, $K \in \mathbb{Z}$.

$\mathrm{Log}(1+\sqrt{3}i) = \ln 4 + i\pi/3$.

(2) $\log 1 = \ln|1| + i\arg 1$
$= 0 + i(0 + 2K\pi) = 2K\pi i$, $K \in \mathbb{Z}$

$\mathrm{Log}\, 1 = 0$ since $\mathrm{Arg}\, 1 = 0$.

(3) $\log -1 = \ln|-1| + i\arg(-1)$
$= 0 + i(\pi + 2K\pi) = (2K+1)\pi i$

$\mathrm{Log} -1 = \pi i$ since $\mathrm{Arg} -1 = \pi$. //

(4) Familiar properties of logarithms from calculus may not hold:

(a) $\mathrm{Log}((-1+i)^2) \neq 2\,\mathrm{Log}(-1+i)$

(b) $\log i^2 \neq 2 \log i$

(a) $\mathrm{Log}(-1+i)^2 = \ln|-1+i|^2 + i\mathrm{Arg}(-1+i)^2$
$= \ln \sqrt{2}^2 + i(-\pi/2)$
$= \ln 2 - i\pi/2$

$2\,\mathrm{Log}(-1+i) = 2(\ln|-1+i| + i\,\mathrm{Arg}(-1+i))$
$= 2\ln\sqrt{2} + 2i\,3\pi/4$
$= \ln 2 + i\,3\pi/2$.

(b) $\log i^2 = \ln|i|^2 + i\arg i^2$
$= 0 + i(\pi + 2K\pi) = i(2K+1)\pi$, $K \in \mathbb{Z}$

$2\log i = 2(\ln|i| + i\arg i)$
$= 2i(\pi/2 + 2K\pi) = i(4K+1)\pi$, $K \in \mathbb{Z}$ //

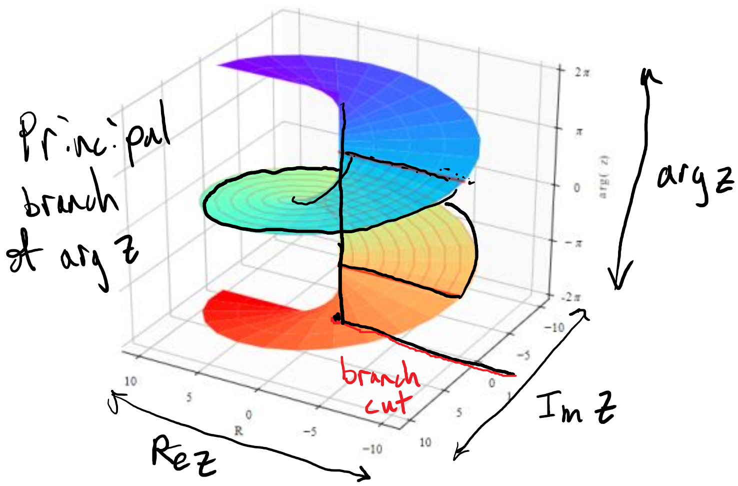
$\boxed{\text{Definition (Branch of a multiple-valued function)}}$ A branch of a multiple-valued function f is a single-valued function F that:

(1) is analytic on some domain D;

(2) assigns to each $z \in D$ precisely one value $F(z)$ of $f(z)$.

A portion of a line or curve in the complex plane is called a branch cut for f if a branch of f is defined on it's complement. A point belonging to every branch cut of f is a branch point. //

$f(z) = \arg z$



$\boxed{\text{Proposition (Branches of } \log z)}$ Let $\alpha \in \mathbb{R}$. The function

$$F(z) = \ln r + i\theta, \quad (r > 0, \ \underline{\alpha < \theta < \alpha + 2\pi})$$

is a branch of $f(z) = \log z$.

Proof. It is clear that $F(z)$ is single-valued and for each z, $F(z)$ is a value of $\log z$. We need to show that F is analytic. Note that $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$ have continuous partial derivatives on the domain of definition.

We have $\quad u_r = \frac{1}{r} \qquad v_r = 0$

$\qquad\qquad u_\theta = 0 \qquad v_\theta = 1.$

Evidently,

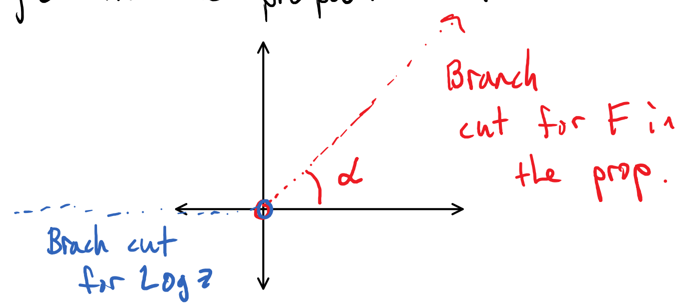
$$r u_r = \frac{r}{r} = 1 = v_\theta$$
$$-r v_r = 0 = u_\theta .$$

So the Cauchy-Riemann eq are satisfied, hence F is analytic.

In fact
$$\boxed{\begin{aligned} \frac{d}{dz} F(z) &= e^{-i\theta}(u_r + i v_r) \\ &= e^{-i\theta}\left(\frac{1}{r}\right) = \frac{1}{z} . \end{aligned}}$$

In particular, $\operatorname{Log} z$ is a branch of $\log z$ and
$$\boxed{\frac{d}{dz} \operatorname{Log} z = \frac{1}{z} .}$$

The branch cut for $\log z$ in the proposition is the ray $r > 0$, $\theta = \alpha$



The branch cut for $\operatorname{Log} z$ is the ray $r > 0$, $\theta = \pi$. The origin is a branch point for $\log z$. //

$\boxed{\text{Proposition}}$ For all $z, w \in \mathbb{C} \setminus \{0\}$,

(1) $\log zw = \log z + \log w$

(2) $\log z/w = \log z - \log w$

These equations are interpreted as follows: given values of two of the logarithms in the equation, there is a value of the third satisfying the eq.

Proof. Compare w/ $\arg zw = \arg z + \arg w$ from Ch 1.

(1) We have
$$\log z + \log w = \ln|z| + i \arg z + \ln|w| + i \arg w$$
$$= \ln|z| + \ln|w| + i(\underbrace{\arg z + \arg w}_{\arg zw})$$

$$= \ln|z||w| + i \arg zw$$
$$= \ln|zw| + i \arg zw$$
$$= \log zw.$$

(2) follows from (1).

$\blacksquare$

The statement does not hold if $\log z$ is replaced w/ $\operatorname{Log} z$.

$\boxed{\text{Example (Integer powers and roots)}}$ The logarithmic function can be used to compute integer powers and roots (as previously defined).

(1) $z^n = e^{n \log z}$, $n \in \mathbb{Z}$

(2) $z^{1/n} = e^{1/n \log z}$, $n \in \mathbb{N}$.

For (1),
$$e^{n\log z} = e^{n(\ln|z| + i\arg z)}$$
$$= e^{n(\ln|z| + i(\operatorname{Arg} z + 2k\pi))}$$
$$= e^{n\ln|z|} e^{ni\operatorname{Arg} z} e^{2nk\pi i}$$
$$= |z|^n e^{i(n\operatorname{Arg} z)} = |z|^n \left(e^{i\operatorname{Arg} z}\right)^n = \left(|z| e^{i\operatorname{Arg} z}\right)^n$$
$$= z^n.$$

polar form of z.

(2)
$$e^{\frac{1}{n}\log z} = e^{\frac{1}{n}(\ln|z| + i\arg z)}$$
$$= e^{\frac{1}{n}(\ln|z| + i(\operatorname{Arg} z + 2k\pi))}$$
$$= \sqrt[n]{|z|}\, e^{i\left(\frac{\operatorname{Arg} z + 2k\pi}{n}\right)} = z^{1/n}.$$

//

Power Functions

$\boxed{\text{Definition (Power function)}}$ The power function z^c for a fixed complex number $c \in \mathbb{C}$ is the multiple-valued function

$$z^c \stackrel{\text{def}}{=} e^{c \log z}, \qquad z \neq 0.$$

//

$\boxed{\text{Proposition (Branches of } z^c)}$ A branch of z^c is determined by specifying a branch of $\log z$:

$$\log z = \ln r + i\theta \qquad (r>0,\ \alpha < \theta < \alpha + 2\pi).$$

Moreover,

$$\frac{d}{dz} z^c = c\, z^{c-1} \qquad (|z|>0,\ \alpha < \arg z < \alpha + 2\pi).$$

Proof. We only need to check that z^c is analytic once a branch of $\log z$ has been specified. Since $z^c = e^{c\log z}$ is the composition of two analytic functions e^z and $c \log z$, z^c is analytic by the chain rule. Moreover,

$$\begin{aligned}
\frac{d}{dz} z^c &= \frac{d}{dz} e^{c\log z} \\
&= e^{c\log z} \cdot \frac{d}{dz}(c\log z) \\
&= \frac{c}{z} e^{c\log z} = c\,\frac{e^{c\log z}}{e^{\log z}} = c\,e^{(c-1)\log z} \\
&= c\,z^{c-1}. \quad \blacksquare
\end{aligned}$$

The principal branch of z^c is defined by specifying the principal branch $\operatorname{Log} z$ of $\log z$. The principal branch of z^c reduces to the usual power function when $z = x \in \mathbb{R}$.

We can define the exponential function with base c by interchanging the roles of z and c.

$\boxed{\text{Definition (exponential function of base } c)}$ The exponential function of base c, $c \neq 0$, is defined via

$$c^z \overset{\text{def}}{=} e^{z\log c}$$

Note: c^z is multiple valued since $\log c$ is. When a value of $\log c$ is specified, c^z is entire and

$$\frac{d}{dz}c^z = \frac{d}{dz}e^{z\log c} = e^{z\log c}\cdot\frac{d}{dz}(z\log c)$$
$$= c^z \log c. \qquad //$$

$\boxed{\text{Question :}}$ what happens if we take $c = e$ (Eulers number)?

Take the principal value $\text{Log}\, e$ in the definition.

$$e^z = e^{z\,\text{Log}\, e} = e^{z(\ln e + i\,\text{Arg}\, e)} = e^{z(1+0)} = e^z. \qquad //$$

$\boxed{\text{Example}}$

(1) Compute $i^i = e^{i\log i} = e^{i(\ln|i| + i\arg i)}$

$$= e^{i^2(\pi/2 + 2k\pi)}, \qquad k\in\mathbb{Z}$$
$$= e^{-\pi/2}\, e^{-2k\pi}, \qquad k\in\mathbb{Z}.$$

(2) Compute $(-1)^{1/\pi} = e^{\frac{1}{\pi}\log(-1)}$

$$= e^{\frac{1}{\pi}(\ln|-1| + i\arg -1)}$$
$$= e^{\frac{1}{\pi}i(\pi + 2k\pi)}, \qquad k\in\mathbb{Z}$$
$$= e^{i(2k+1)}, \qquad k\in\mathbb{Z}.$$

//

Trigonometric Functions

Recall, for any $z \in \mathbb{C}$,

$$\operatorname{Re} z = \frac{z + \bar{z}}{2} \qquad \operatorname{Im} z = \frac{z - \bar{z}}{2i}.$$

Hence, for $x \in \mathbb{R}$,

$$\cos x = \operatorname{Re}(e^{ix}) = \frac{e^{ix} + \overline{e^{ix}}}{2} = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \operatorname{Im}(e^{ix}) = \frac{e^{ix} - \overline{e^{ix}}}{2i} = \frac{e^{ix} - e^{-ix}}{2i}.$$

This suggests a way to extend the domain of definition of the sine and cosine functions to all of $\mathbb{C}$.

Definition (sine and cosine) The sine and cosine functions of a complex variable z are defined via

$$\sin z \overset{\text{def}}{=} \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z \overset{\text{def}}{=} \frac{e^{iz} + e^{-iz}}{2}$$

//

By our calculation above, $\sin z$ and $\cos z$ reduce to the ordinary sine and cosine functions when z is real.

Proposition (Analyticity of sine and cosine)

(1) $\sin z$ and $\cos z$ are entire

(2) $\frac{d}{dz} \sin z = \cos z$ and $\frac{d}{dz} \cos z = -\sin z$

Proof. (1) $\sin z$ / $\cos z$ are entire since they are linear combinations of entire functions e^{iz}, e^{-iz}.

(2) $$\frac{d}{dz}\sin z = \frac{d}{dz}\frac{e^{iz}-e^{-iz}}{2i} = \frac{1}{2i}\frac{d}{dz}(e^{iz}-e^{-iz})$$
$$= \frac{1}{2i}\left(ie^{iz}+ie^{-iz}\right)$$
$$= \frac{e^{iz}+e^{-iz}}{2} = \cos z.$$

$$\frac{d}{dz}\cos z = \frac{1}{2}\frac{d}{dz}e^{iz}+e^{-iz} = \frac{1}{2}\left(ie^{iz}-ie^{-iz}\right)$$
$$= i\left(\frac{e^{iz}-e^{-iz}}{2}\right) = -\left(\frac{e^{iz}-e^{-iz}}{2i}\right)$$
$$= -\sin z. \quad \blacksquare$$

Various identities hold. Here are a few:

(1) $\sin -z = -\sin z$

(2) $\cos -z = \cos z$

(3) $\sin z+w = \sin z\cos w + \cos z\sin w$

(4) $\cos z+w = \cos z\cos w - \sin z\sin w$

(5) $\sin 2z = 2\sin z\cos z$

(6) $\cos 2z = \cos^2 z - \sin^2 z$

(7) $\sin^2 z + \cos^2 z = 1$

(8) $\sin(z+2\pi) = \sin z$

(9) $\cos(z+2\pi) = \cos z$

(10) $\sin(z+\pi/2) = \cos z$

(11) $\sin(z-\pi/2) = -\cos z$

//

To define the other trig functions, we need to understand the zeros of $\sin z$, $\cos z$. To do this, we need the following new identities:

Proposition

(1) $\sin(iy) = i\sinh y$ and $\cos(iy) = \cosh y$

(2) $\sin z = \sin x\cosh y + i\cos x\sinh y$

$\cos z = \cos x\cosh y - i\sin x\sinh y$

(3) $|\sin z|^2 = \sin^2 x + \sinh^2 y$

$|\cos z|^2 = \cos^2 x + \sinh^2 y$

Recall:

$$\sinh y = \frac{e^y - e^{-y}}{2}$$

$$\cosh y = \frac{e^y + e^{-y}}{2}$$

Proof.

(1) $$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i} = i\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= i\sinh y.$$

$$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y} + e^{y}}{2} = \cosh y.$$

(2) Write $z = x + iy$. Then

$$\sin z = \sin x + iy \overset{(2)}{\underset{\text{Part (1)}}{=}} \sin x\cos iy + \cos x\sin iy$$

$$= \underbrace{\sin x\cosh y}_{u} + i\underbrace{\cos x\sinh y}_{v}.$$

Then $$\cos z = \frac{d}{dz}\sin z$$

$$= u_x + iv_x = \cos x\cosh y - i\sin x\sinh y.$$

(3) $$|\sin z|^2 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y$$

$$= \sin^2 x\cosh^2 y - \sin^2 x\sinh^2 y + \sin^2 x\sinh^2 y + \cos^2 x\sinh^2 y$$

$$= \sin^2 x\underbrace{(\cosh^2 y - \sinh^2 y)}_{=1} + \sinh^2 y\underbrace{(\sin^2 x + \cos^2 x)}_{=1}$$

$$= \sin^2 x + \sinh^2 y. \quad \blacksquare$$